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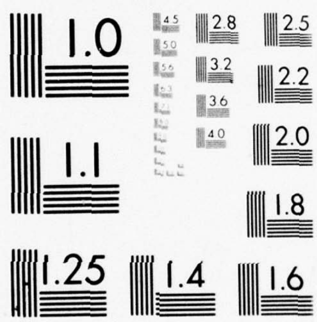

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TECHNICAL REPORT T-78-8

ACCURATE REDUCED MODEL REPRESENTATION AND STABILITY DETERMINATION OF T6 MISSILE SYSTEM

R. E. Yates and L. S. Shieh
Guidance and Control Directorate
Technology Laboratory

U.S. ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND

14 October 1977

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A method, based on a model reduction technique, is developed for accurately representing the continuous-time state equations of T6 missiles by discrete-time state equations. By using the accurate models proposed, the relaxation of the sampling period requirement may be accomplished which facilitates the practical implementation of these systems. Also, the

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stability of coupled multivariable systems such as the yaw and roll subsystems of the T6 missile is investigated. The well-known Routh stability criterion for a single variable system has been partially extended to the matrix Routh criterion for these type multivariable systems.

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1.0 INTRODUCTION

The T6 missile is a small semi-active terminal homing air-to-ground missile. It consists of many subsystems such as missile airframe dynamics, fin actuator dynamics, attitude sensors, geometric models, laser seeker dynamics, stabilization filters, and guidance filters. The mathematical description of this complex system results in time-varying, coupled high order, nonlinear differential equations. A six degree-of-freedom program has been written for the simulation of this system. The system is currently modeled by high order, coupled linear scalar differential equations or high degree transfer functions in the s domain. The stabilization filter has been designed to smooth the response of the missile and the guidance filter has been designed to guide the missile for accurate impact of a target.

Several performance studies of the missile system have been made recently. For example, the continuous-time linear model of the missile system was converted to a discrete-time model[1,2] so that a digital stabilization filter and a digital guidance filter could be synthesized. Another example is an improved trajectory which was designed by using modern dual-model control law theory[3] so that the missile impacts a target at a predetermined pitch attitude approach angle.

In reviewing previous studies, several research problems have been identified by the authors for improving these prior results. In this report we will concentrate on two subjects: (1) accurate representation of continuous-time state equations of T6 missile system by discrete-time

state equations and (2) stability of coupled multivariable missile systems. These two subjects were chosen due to the possibility for immediate, practical implementation and theoretical analysis. For example, the missile system is currently represented by high order continuous state equations. These state equations need to be converted to discrete-time state equations from which the implementations of the controllers for the missile system can be accomplished by microprocessors. In the prior results, a crude approximate discrete-time model has been used for the analysis and synthesis of digital autopilot systems. Accurate simulation and control of the missile system require that a more accurate model be used. In this report we give an approach, which is based on a model reduction technique developed by the authors, for the formulation of accurate discrete-time models.

The T6 missile yaw and roll dynamics are heavily coupled multivariable subsystems. A primary and often vexing concern in the design of coupled multivariable systems is the determination of the stability of the coupled systems. Since the methods for determining the stability of coupled multivariable systems are quite different than those developed for uncoupled systems, an approach for determining the stability of the coupled systems has been developed and is presented in this report.

2.0 ACCURATE REPRESENTATION OF CONTINUOUS-TIME STATE EQUATIONS OF T6 MISSILE SYSTEM BY DISCRETE-TIME STATE EQUATIONS

2.1 INTRODUCTION

The accurate description of many practical systems results in high order continuous-time state equations which are difficult to work with -- whether one is simulating, realizing or designing these systems. In particular, the systems which consist of time-varying and/or time-delay components are especially difficult. However, the difficulty of these complex processes can often be significantly reduced if a set of continuous-time state equations can be accurately represented by a discrete-time state equation set. This simplification can be accomplished by several methods.

A procedure often used for small, time-invariant systems is to directly evaluate the state transition matrix and its convolution with the input. This yields an exact representation of a discrete-time state equation. However, for either a large system, a time-varying system, or a system with slight variations of system parameters, this approach is not practical. Tustin[4] and Boxer and Thaler[5] have indirectly constructed approximate discrete-time state equations via s- or z-transforms. In practical applications, it is often necessary to determine directly an approximate model in the time domain. Recently, Shieh, et al[6] suggested a time domain method for this problem by using the newly developed block-pulse functions (developed from Walsh functions[7]). In this report, it is shown that the models obtained by the referenced authors[4,5,6] are equivalent and that this model is one possible model resulting from the use of the method presented here. A convenient method[2] often used in

industry is to construct an approximate discrete-time state equation by truncating the infinite series obtained from expansion of the exact state transition matrix and its convolution with the input in the time domain. However, the truncating error depends heavily on the number of terms and the sampling period used. Based on the idea of multi-feedback and multi-feedforward control theory, a method for determination of the approximate discrete-time state equations is presented in this report. Matrix continued fractions[8] are used as a basis for this investigation.

2.2 APPROXIMATION OF STATE TRANSITION MATRIX

Consider the system governed by the continuous-time state equations,

$$\dot{x}_o(t) = Ax_o(t) + Bu_o(t)$$

$$x_o(0) = x_o \tag{1}$$

where A and B are n X n and n X p constant matrices, respectively, $x_o(t)$ is an n X 1 column state vector, $u_o(t)$ is a p X 1 continuous-time input vector, and $x_o(0)$ is an initial vector. The solution of Eq. (1) is

$$x_o(t) = \Phi(t)\ x_o(0) + \int_0^t \Phi(t-\lambda)\ Bu_o(\lambda)\ d\lambda \tag{2a}$$

where $\Phi(t)$ is the continuous-time state transition matrix, or

$$\Phi(t) = e^{At} = [e^{AT}]^k = [\Phi(T)]^k \tag{2b}$$

and t = kT, k is an integer. If $\Phi(t)$ and the convolution integral in Eq. (2a) can be explicitly evaluated, the corresponding discrete-time state equation is obtained by substituting t = kT into Eq. (2a). From the point of view of practical implementation, we are interested in the

system in which the input signal $u_o(t)$ is a piecewise-constant and the sampling instants are 0, T, 2T, ..., which gives us a rectangular approximation of $u_o(t)$. This is equivalent to inserting a sample-and-zero-order-hold device before an integrator. The approximate input is designated as u(t), or

$$u(t) = u(kT) \cong u_o(t) \text{ for } kT \leq t < (k+1)T \tag{3}$$

The approximate state due to u(t) is designated as x(t). The new state equations are

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$x(0) = x_o \tag{4a}$$

The solution of Eq. (4a) is

$$x(kT + T) = e^{AT}\, x(kT) + \int_0^T e^{A\alpha}\, B\, d\alpha \cdot u(kT) \tag{4b}$$

where $\alpha = T - \lambda$. Letting x(kT) = x(k) and u(kT) = u(k), we have the discrete-time state equation

$$x(k + 1) = \Phi(T)\, x(k) + Lu(k) \tag{4c}$$

where

$$\Phi(T) = e^{AT}$$

and

$$L = \int_0^T e^{A\alpha}\, B\, d\alpha$$

The solution of Eq. (4c) is

$$x(k) = \Phi(T)^k x_o + \sum_{j=0}^{k-1} \Phi(k-j-1)\ Lu(j) \tag{4d}$$

The system matrix $\Phi(T)$ and its convolution with the input can be expressed by the infinite series:

$$\Phi(T) = e^{AT} = I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \frac{1}{4!}(AT)^4 + \ldots$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!}(AT)^j \tag{5a}$$

and

$$L = \int_0^T e^{A\alpha}\ B\ d\alpha = T[I + \frac{1}{2!} AT + \frac{1}{3!}(AT)^2 + \frac{1}{4!}(AT)^3 + \ldots]B$$

$$= T \sum_{j=0}^{\infty} \frac{1}{(j+1)!} (AT)^j\ B \tag{5b}$$

where I is the identity matrix. Eq. (5b) can be further simplified as

$$L = T[I + AT + \frac{1}{2!}(AT)^2 + \ldots - I](AT)^{-1}\ B$$

$$= [e^{AT} - I]A^{-1}\ B = [\Phi(T) - I]A^{-1}\ B \tag{5c}$$

For practical reasons, we are interested in the approximate $\Phi(T)$, denoted as $\Phi^*(T)$ or G. Eq. (4c) becomes

$$x^*(k + 1) = G\ x^*(k) + Mu(k) \tag{6a}$$

where

$$G = \Phi^*(T) \cong \Phi(T)$$

$$M = [G - I]A^{-1} B \cong L$$

$$x^*(k) \cong x(k) \tag{6b}$$

The solution of Eq. (6a) is

$$x^*(k) = \Phi^*(k)\ x^*(0) + \sum_{j=0}^{k-1} \Phi^*(k-j-1)\ Mu(j) \tag{6c}$$

where

$$x^*(0) = x(0)$$
$$\Phi^*(k) = \Phi^*(kT) = G^k$$

where G^k is the approximate discrete-time state transition matrix.

The natural question arises -- how to accurately determine G and M? A popular method often used in industry[2] is to approximate $\Phi(T)$ in Eq. (5a) by truncating the infinite series:

$$\Phi(T) = I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \frac{1}{4!}(AT)^4 + \ldots \tag{7a}$$

$$\cong I + AT \tag{7b}$$

$$\cong I + AT + \frac{1}{2!}(AT)^2 \tag{7c}$$

$$\cong I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 \tag{7d}$$

$$\cong I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \frac{1}{4!}(AT)^4 \tag{7e}$$

$$\cong \ldots$$

If a sufficiently large number of terms in Eq. (7a) is used, then a satisfactory approximation may be obtained. In this report, we introduce a geometric series to approximate the infinite series in Eq. (7a) as follows:

$$\Phi(T) = I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \frac{1}{4!}(AT)^4 + \frac{1}{5!}(AT)^5 + \ldots \quad (8a)$$

$$= I + AT + \frac{1}{2}(AT)^2 + \sum_{j=3}^{\infty} \frac{1}{j!}(AT)^j \quad (8b)$$

$$\cong I + AT + \frac{1}{2}(AT)^2 + \frac{1}{2^2}(AT)^3 + \frac{1}{2^3}(AT)^4 + \frac{1}{2^4}(AT)^5 + \ldots \quad (8c)$$

$$= I + AT + \frac{1}{2}(AT)^2 + \sum_{j=3}^{\infty} \frac{1}{2^{j-1}}(AT)^j \quad (8d)$$

$$= I + [I + \frac{1}{2} AT + \frac{1}{2^2}(AT)^2 + \frac{1}{2^3}(AT)^3 + \ldots]AT \quad (8e)$$

$$= I + [I - \frac{1}{2} AT]^{-1} AT \quad (8f)$$

$$= [I - \frac{1}{2} AT]^{-1}[I + \frac{1}{2} AT] \quad (8g)$$

$$\triangleq \Phi^*(T) \quad (8h)$$

The infinite series in the brackets of Eq. (8e) is the well-known geometric series. Comparing Equations (7c) and (8d) with Eq. (8b), we observe that the first three terms in all three equations are identical, while other terms differ by their weighting factors: zero in Eq. (7c), $\frac{1}{2^{j-1}}$ in Eq. (8d) and $\frac{1}{j!} = \frac{1}{j(j-1)(j-2)\ \ldots\ 1}$ in Eq. (8b). The approximation of $\Phi(T)$ in Eq. (8a) given by $\Phi^*(T)$ in Eq. (8h) is much better than that of Eq. (7c).

For elaboration of the above conclusion, it is convenient to investigate the inverse system of $\Phi^*(T)$, denoted by $\Phi^*(-T)$. The formulation of $\Phi^*(-T)$ is

$$\Phi^*(-T) = [I + AT + \frac{1}{2}(AT)^2 + \sum_{j=3}^{\infty} \frac{1}{2^{j-1}}(AT)^j]^{-1}$$

$$= [I + [I - \frac{1}{2} AT]^{-1} AT]^{-1}$$

$$= [I + [(AT)^{-1} + (- \frac{1}{2}I)]^{-1}]^{-1}$$

$$= [H_1 + [H_2 + [H_3]^{-1}]^{-1}]^{-1} \tag{9a}$$

where $H_1 = I$, $H_2 = (AT)^{-1}$ and $H_3 = -2I$.

Equation (9a) is the formulation of a matrix continued fraction[8] which is considered as a generalized multi-feedback, multi-feedforward control system. A representative control system for Eq. (9a) is shown in Figure 1. Note that H_1 is a feedback matrix gain, while H_2 and H_3^{-1} are the parallel feedforward matrix gains. The use of H_3 is very important, for example, if H_3 is neglected in Eq. (9a), we have

$$\Phi^*(-T) = [H_1 + [H_2]^{-1}]^{-1}$$

$$= [I + AT]^{-1} \tag{9b}$$

or

$$\Phi^*(T) = I + AT \tag{9c}$$

Equation (9c) is equal to Eq. (7b), which has the first two terms of the infinite series in Eq. (7a). Therefore H_3 not only contributes the third term, $\frac{1}{2}(AT)^2$, but also generates an important series, $\sum_{j=3}^{\infty} \frac{1}{2^{j-1}}(AT)^j$, as shown in Eq. (9a). From this we conclude that if we can establish more terms in the inner loop of the control system in

Figure 2-1, we may have a more accurate approximation. For example, if we insert a constant matrix $H_4 = -3(AT)^{-1}$ in Eq. (9a), we obtain

$$\begin{aligned}
\Phi^*(-T) &\cong [H_1 + [H_2 + [H_3 + [H_4]^{-1}]^{-1}]^{-1}]^{-1} \\
&= [H_2H_3H_4 + (H_2 + H_4)][H_1H_2H_3H_4 + (H_1H_2 + H_1H_4 + H_3H_4) + I]^{-1} \\
&= [6(AT)^{-2} - 2(AT)^{-1}][6(AT)^{-2} + 4(AT)^{-1} + I]^{-1} \\
&= [I - \frac{1}{3}AT][I + \frac{2}{3}AT + \frac{1}{6}(AT)^2]^{-1} \\
&= \{[I - \frac{1}{3}AT]^{-1}[I + \frac{2}{3}AT + \frac{1}{6}(AT)^2]\}^{-1} \\
&= \{I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \frac{1}{(1.5)(4!)}[I + \frac{1}{3}AT \\
&\quad + \frac{1}{3^2}(AT)^2 + \frac{1}{3^3}(AT)^3 + \dots](AT)^4\}^{-1} \\
&= \{I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \frac{1}{(1.5)(4!)}[I - \frac{1}{3}AT]^{-1}(AT)^4\}^{-1} \qquad (9d)
\end{aligned}$$

or

$$\begin{aligned}
\Phi^*(T) &= [I - \frac{1}{3}AT]^{-1}[I + \frac{2}{3}AT + \frac{1}{6}(AT)^2] \\
&= I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \frac{1}{(1.5)(4!)}[I - \frac{1}{3}AT]^{-1}(AT)^4 \qquad (9e)
\end{aligned}$$

A block diagram corresponding to Eq. (9d) is shown in Figure 2-2. H_1 and H_3 are multi-feedback gains; H_2 and H_4 are multi-feedforward gains. H_4 contributes one more important term, $\frac{1}{3!}(AT)^3$, than that of H_3 in Eq. (9a) and another geometric series, $(I - \frac{1}{3}AT)^{-1}$, with the weighting factor $\frac{1}{(1.5)(4!)}(AT)^4$. Comparing Eq. (9e) with Eq. (7d), it can be seen that Eq. (9e) gives a better approximation. These $H_1 \dots H_4$ can be generated by performing the matrix continued fraction expansion[8] on $\Phi(T)$ given in Eq. (8a).

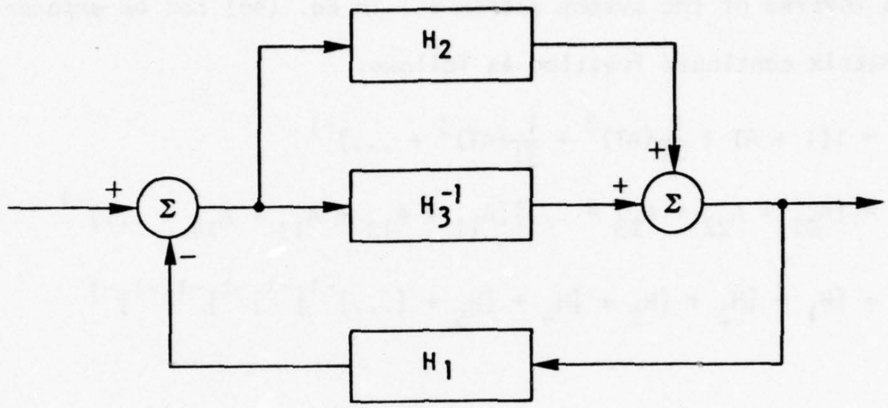

FIGURE 2-1. BLOCK DIAGRAM FOR $\Phi^*(-T) = [H_1 + [H_2 + [H_3]^{-1}]^{-1}]^{-1}$

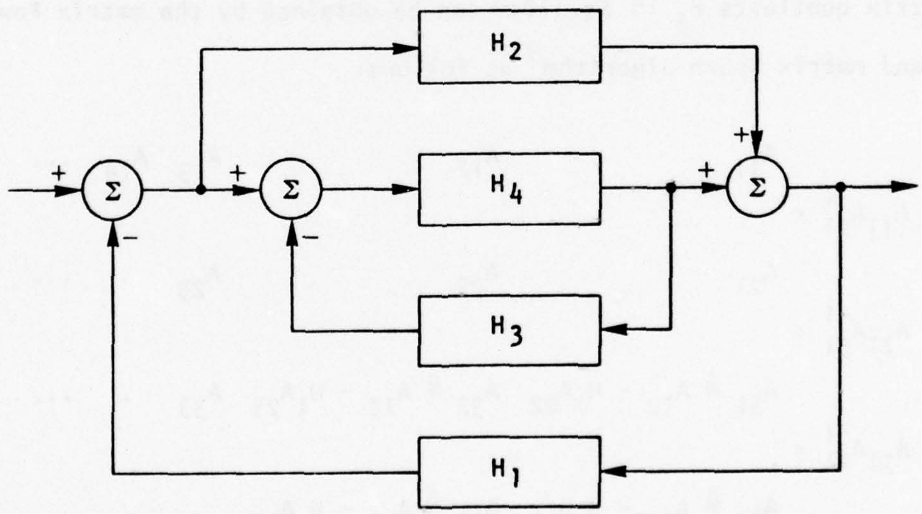

FIGURE 2-2. BLOCK DIAGRAM FOR $\Phi^*(-T) = [H_1 + [H_2 + [H_3 + [H_4]^{-1}]^{-1}]^{-1}]^{-1}$

2.3 APPROXIMATION BY MATRIX CONTINUED FRACTIONS

The inverse of the system matrix e^{AT} in Eq. (4c) can be expanded into a matrix continued fraction as follows:

$$e^{-AT} = I[I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \ldots]^{-1} \tag{10a}$$

$$= [A_{21} + A_{22} + A_{23} + \ldots][A_{11} + A_{12} + A_{13} + A_{14} + \ldots]^{-1} \tag{10b}$$

$$= [H_1 + [H_2 + [H_3 + [H_4 + [H_5 + [\ldots]^{-1}]^{-1}]^{-1}]^{-1}]^{-1}]^{-1} \tag{10c}$$

where

$$A_{21} = I,\ A_{2,j} = 0 \text{ for } j = 2,3,\ldots$$

$$A_{11} = I,\ A_{1,j} = \frac{1}{(j-1)!}(AT)^{j-1} \text{ for } j = 2,3,\ldots$$

The matrix quotients H_j in Eq. (10c) can be obtained by the matrix Routh array and matrix Routh algorithm[9] as follows:

$$\begin{array}{lllll}
 & A_{11} & A_{12} & A_{13}\quad A_{14} & \cdots \\
H_1 = A_{11}A_{21}^{-1} < & & & & \\
 & A_{21} & A_{22} & A_{23}\quad . & \cdots \\
H_2 = A_{21}A_{31}^{-1} < & & & & \\
 & A_{31} \triangleq A_{12} - H_1A_{22} & A_{32} \triangleq A_{13} - H_1A_{23} & A_{33}\quad . & \cdots \\
H_3 = A_{31}A_{41}^{-1} < & & & & \\
 & A_{41} \triangleq A_{22} - H_2A_{32} & A_{42} \triangleq A_{23} - H_2A_{33} & . \quad . & \\
\quad . \quad < & & & & \\
\quad . & . & .. & &
\end{array} \tag{11a}$$

The block elements of the first and second row of Eq. (11a) are the matrices in Eq. (10b). The block elements of the subsequent rows are evaluated by the following matrix Routh algorithm:

$$H_p = A_{p,1}A_{p+1,1}^{-1} \text{ for } p = 1,2,\ldots$$

$$A_{j,\ell} = A_{j-2,\ell+1} - H_{j-2}A_{j-1,\ell+1} \text{ for } \ell = 1,2,\ldots;\ j = 3,4,\ldots \tag{11b}$$

$$\text{rank}(A_{p+1,1}) = n$$

The H_j so obtained are

$$H_1 = I,\ H_2 = (AT)^{-1},\ H_3 = -2I,\ H_4 = -3(AT)^{-1}$$

$$H_5 = 2I,\ H_6 = 5(AT)^{-1},\ H_7 = -2I,\ H_8 = -7(AT)^{-1}$$

$$\ldots$$

$$H_j = 2I,\ H_{j+1} = j(AT)^{-1},\ H_{j+2} = -2I,\ H_{j+3} = -(j+2)(AT)^{-1}$$

$$\text{for } j = 5,\ 9,\ 13,\ 17,\ \ldots,\ (\ell + 4),\ \ldots \tag{11c}$$

The approximation of Eq. (10) can be obtained by truncating the matrices in the inner positions of Eq. (10c); i.e.,

$$\begin{aligned} e^{-AT} &\cong [H_1 + [H_2 + [H_3 + [H_4 + [H_5 + [\ldots]^{-1}]^{-1}]^{-1}]^{-1}]^{-1}]^{-1} \\ &\cong [H_1 + [H_2]^{-1}]^{-1} \\ &\cong [H_1 + [H_2 + [H_3]^{-1}]^{-1}]^{-1} \\ &\cong [H_1 + [H_2 + [H_3 + [H_4]^{-1}]^{-1}]^{-1}]^{-1} \\ &\cong \ldots \end{aligned} \tag{12}$$

By successively determining the inversions of the matrices H_j in Eq. (12), we can obtain the approximation in Eq. (12). In order to avoid these tedious inversion operations, the following technique[10] may be used to simplify the matrix inversions.

To demonstrate the procedure, we choose $n = 5$ with H_j, $j = 1 \ldots n$ given. First, we formulate a chain of n matrices,

$$P = \prod_{j=1}^{n} P_j = P_1P_2P_3P_4P_5 \tag{13a}$$

$$= \begin{bmatrix} H_1 & I & & & \\ & H_2 & I & & \\ & & H_3 & I & \\ & & & H_4 & I \\ & & & & H_5 \end{bmatrix} \begin{bmatrix} H_2 & I & & & \\ & H_3 & I & & \\ & & H_4 & I & \\ & & & H_5 & \\ & & & & I \end{bmatrix} \begin{bmatrix} H_3 & I & & & \\ & H_4 & I & & \\ & & H_5 & & \\ & & & I & \\ & & & & I \end{bmatrix} \begin{bmatrix} H_4 & I & & & \\ & H_5 & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{bmatrix} \begin{bmatrix} H_5 & & & & \\ & I & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{bmatrix} \tag{13b}$$

The structure of P is as follows: The matrix quotients H_j are placed on the diagonal in ascending order; the first matrix P_1 starts with H_1 and the next, P_2 with H_2, etc. Then identity matrices are placed on the location above the H_j and on the diagonal as shown in Eq. (13b). All other block elements are zero. The product of the chain of the matrices in Eq. (13a) is

$$P = \begin{bmatrix} (H_1H_2H_3H_4H_5) & (H_1H_2H_3 + H_1H_2H_5 + H_1H_4H_5 + H_3H_4H_5) & (H_1 + H_3 + H_5) & 0 & 0 \\ 0 & (H_2H_3H_4H_5) & (H_2H_3 + H_2H_5 + H_4H_5) & I & 0 \\ 0 & 0 & (H_3H_4H_5) & (H_3 + H_5) & 0 \\ 0 & 0 & 0 & (H_4H_5) & I \\ 0 & 0 & 0 & 0 & (H_5) \end{bmatrix} \tag{13c}$$

The approximation of e^{-AT} can be obtained from the block elements in the first two rows of P, or

$$
\begin{aligned}
e^{-AT} &\cong [H_1 + [H_2 + [H_3 + [H_4 + [H_5]^{-1}]^{-1}]^{-1}]^{-1}]^{-1} \\
&= [H_2H_3H_4H_5 + (H_2H_3 + H_2H_5 + H_4H_5) + I] \cdot \\
&\quad [H_1H_2H_3H_4H_5 + (H_1H_2H_3 + H_1H_2H_5 + H_1H_4H_5 + H_3H_4H_5) + (H_1 + H_3 + H_5)]^{-1} \\
&= [\sum_{j=1}^{n} P_{2,j}][\sum_{j=1}^{n} P_{1,j}]^{-1} = [\sum_{j=1}^{n} P_{2,j}\, P_{11}^{-1}][\sum_{j=1}^{n} P_{1,j}\, P_{11}^{-1}]^{-1} \\
&= [\sum_{j=1}^{n} P_{1,j}\, P_{11}^{-1}]^{-1}[\sum_{j=1}^{n} P_{2,j}\, P_{11}^{-1}]
\end{aligned} \tag{14}
$$

where $P_{1,j}$ and $P_{2,j}$ are the block elements in the first and second rows of the matrix P, respectively.

Using the above procedure to determine $P_{1,j}$ and $P_{2,j}$, and substituting the values of H_j, yields

$$e^{AT} = H_1 +[H_2 + [H_3 + [H_4 + [\ldots]^{-1}]^{-1}]^{-1}]^{-1} \tag{15a}$$

$$= [\sum_{j=1}^{n=\infty} P_{2,j}\, P_{11}^{-1}]^{-1}[\sum_{j=1}^{n=\infty} P_{1,j}\, P_{11}^{-1}] \tag{15b}$$

$$\cong I + AT \tag{15c}$$

$$\cong [I - \frac{1}{2} AT]^{-1}[I + \frac{1}{2} AT] \tag{15d}$$

$$\cong [I - \frac{1}{3} AT]^{-1}[I + \frac{2}{3} AT + \frac{1}{6}(AT)^2] \tag{15e}$$

$$\cong [I - \frac{1}{2} AT + \frac{1}{12}(AT)^2]^{-1}[I + \frac{1}{2} AT + \frac{1}{12}(AT)^2] \tag{15f}$$

$$\cong [I - \frac{2}{5} AT + \frac{1}{20}(AT)^2]^{-1}[I + \frac{3}{5} AT + \frac{3}{20}(AT)^2 + \frac{1}{60}(AT)^3] \tag{15g}$$

$$\cong \ldots$$

Equations (15d) and (15e) are equal to Eq. (8g) and Eq. (9e), respectively.

2.4 DERIVATION OF DISCRETE-TIME STATE EQUATIONS

Rewriting Eq. (6a), we have

$$x^*(k + 1) = Gx^*(k) + Mu(k) \tag{16a}$$

where

$$G = \Phi^*(T) \cong \Phi(T) \tag{16b}$$

$$M = [G - I]A^{-1} B \cong [\Phi(T) - I]A^{-1} B \tag{16c}$$

$$u(k) = u(kT) \cong u_o(kT) \tag{16d}$$

$$x^*(k) \cong x(k) \cong x_o(kT) \tag{16e}$$

The solution of Eq. (16a) is

$$x^*(k) = G^k x^*(0) + \sum_{j=0}^{k-1} G^{k-j-1} Mu(j) \tag{17}$$

The required system matrix G obtained in Eq. (15) is

$$
\begin{aligned}
G &= I + AT \triangleq G_2 \\
&\cong [I - \tfrac{1}{2} AT]^{-1}[I + \tfrac{1}{2} AT] \triangleq G_3 \\
&\cong [I - \tfrac{1}{3} AT]^{-1}[I + \tfrac{2}{3} AT + \tfrac{1}{6}(AT)^2] \triangleq G_4 \\
&\cong [I - \tfrac{1}{2} AT + \tfrac{1}{12}(AT)^2]^{-1}[I + \tfrac{1}{2} AT + \tfrac{1}{12}(AT)^2] \triangleq G_5 \\
&\cong [I - \tfrac{2}{5} AT + \tfrac{1}{20}(AT)^2]^{-1}[I + \tfrac{3}{5} AT + \tfrac{3}{20}(AT)^2 + \tfrac{1}{60}(AT)^3] \triangleq G_6 \\
&\cong [I - \tfrac{1}{2} AT + \tfrac{1}{10}(AT)^2 - \tfrac{1}{120}(AT)^3]^{-1}[I + \tfrac{1}{2} AT + \tfrac{1}{10}(AT)^2 + \tfrac{1}{120}(AT)^3] \triangleq G_7 \\
&\cong [I - \tfrac{3}{7} AT + \tfrac{1}{14}(AT)^2 - \tfrac{1}{210}(AT)^3]^{-1} \cdot \\
&\quad [I + \tfrac{4}{7} AT + \tfrac{1}{7}(AT)^2 + \tfrac{2}{105}(AT)^3 + \tfrac{1}{840}(AT)^4] \triangleq G_8 \\
&\cong [I - \tfrac{1}{2} AT + \tfrac{3}{28}(AT)^2 - \tfrac{1}{84}(AT)^3 + \tfrac{1}{1680}(AT)^4]^{-1} \cdot \\
&\quad [I + \tfrac{1}{2} AT + \tfrac{3}{28}(AT)^2 + \tfrac{1}{84}(AT)^3 + \tfrac{1}{1680}(AT)^4] \triangleq G_9 \\
&\cong [I - \tfrac{4}{9} AT + \tfrac{1}{12}(AT)^2 - \tfrac{1}{126}(AT)^3 + \tfrac{1}{3024}(AT)^4]^{-1} \cdot \\
&\quad [I + \tfrac{5}{9} AT + \tfrac{5}{36}(AT)^2 + \tfrac{5}{252}(AT)^3 + \tfrac{5}{3024}(AT)^4 + \tfrac{1}{15120}(AT)^5] \triangleq G_{10} \\
&\cong [I - \tfrac{1}{2} AT + \tfrac{1}{9}(AT)^2 - \tfrac{1}{72}(AT)^3 + \tfrac{1}{1008}(AT)^4 - \tfrac{1}{3024}(AT)^5]^{-1} \cdot \\
&\quad [I + \tfrac{1}{2} AT + \tfrac{1}{9}(AT)^2 + \tfrac{1}{72}(AT)^3 + \tfrac{1}{1008}(AT)^4 + \tfrac{1}{3024}(AT)^5] \triangleq G_{11} \\
&\cong [I - \tfrac{5}{11} AT + \tfrac{1}{11}(AT)^2 - \tfrac{1}{99}(AT)^3 + \tfrac{1}{1584}(AT)^4 - \tfrac{1}{55440}(AT)^5]^{-1} \cdot \\
&\quad [I + \tfrac{6}{11} AT + \tfrac{3}{22}(AT)^2 + \tfrac{2}{99}(AT)^3 + \tfrac{1}{528}(AT)^4 + \tfrac{1}{9240}(AT)^5 + \tfrac{1}{332640}(AT)^6] \triangleq G_{12} \\
&\cong [I - \tfrac{1}{2} AT + \tfrac{5}{44}(AT)^2 - \tfrac{1}{66}(AT)^3 + \tfrac{1}{792}(AT)^4 - \tfrac{1}{15840}(AT)^5 + \tfrac{1}{665280}(AT)^6]^{-1} \cdot \\
&\quad [I + \tfrac{1}{2} AT + \tfrac{5}{44}(AT)^2 + \tfrac{1}{66}(AT)^3 + \tfrac{1}{792}(AT)^4 + \tfrac{1}{15840}(AT)^5 + \tfrac{1}{665280}(AT)^6] \triangleq G_{13} \\
&\cong \ldots
\end{aligned}
\tag{18}
$$

The subscript of G in Eq. (18) indicates the number of H_j used. Substituting G obtained in Eq. (16c) yields

$$
\begin{aligned}
M &= [G - I]A^{-1} B \\
&\cong TB \triangleq M_2 \\
&\cong T[I - \tfrac{1}{2} AT]^{-1} B \triangleq M_3 \\
&\cong T[I - \tfrac{1}{3} AT]^{-1}[I + \tfrac{1}{6} AT]B \triangleq M_4 \\
&\cong T[I - \tfrac{1}{2} AT + \tfrac{1}{12}(AT)^2]^{-1}B \triangleq M_5 \\
&\cong T[I - \tfrac{2}{5} AT + \tfrac{1}{20}(AT)^2]^{-1}[I + \tfrac{1}{10} AT + \tfrac{1}{60}(AT)^2]B \triangleq M_6 \\
&\cong T[I - \tfrac{1}{2} AT + \tfrac{1}{10}(AT)^2 - \tfrac{1}{120}(AT)^3]^{-1}[I + \tfrac{1}{60}(AT)^2]B \triangleq M_7 \\
&\cong \ldots
\end{aligned} \tag{19}
$$

The subscript of M indicates the number of H_j used.

If the polygonal hold (which is a device to integrate by the trapezoidal approximation[11]) can be realized, a more accurate discrete-time state equation can be obtained. The staircase input $u^*(k)$ is

$$u^*(k) = \frac{u(k+1) + u(k)}{2} \tag{20}$$

The approximate discrete-time state equation is

$$x^*(k + 1) = G\, x^*(k) + Mu^*(k) \tag{21}$$

where G and M are shown in Equations (18) and (19), respectively.

Since more accurate inputs are used in Eq. (21), the response $x^*(k)$ in Eq. (21) is more accurate than the $x^*(k)$ in Eq. (16). If $j = 3$ and H_1, H_2 and H_3 are used to approximate e^{AT}, then G_3 and M_3 should be used. Equation (21) then becomes

$$x^*(k+1) = [I - \frac{1}{2}AT]^{-1}[I + \frac{1}{2}AT]x^*(k) + T[I - \frac{1}{2}AT]^{-1}Bu^*(k) \tag{22}$$

The model given in Eq. (22) has been derived by others[4-6]. Therefore, it is seen that their result is a special case of the results proposed in this paper.

In the z- or s-domain, Eq. (22) can be derived by using the transfer function of an approximate numerical differentiator or integrator[11]:

$$Z[\frac{d}{dt}] \cong \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \cong s \tag{23a}$$

$$Z[\int dt] \cong \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \cong s^{-1} \tag{23b}$$

In the frequency domain, the numerical integration operator in Eq. (23b) is

$$T(j\omega) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}\Bigg|_{z=e^{j\omega t}} = -j\,\frac{T}{2}\cot\frac{\omega T}{2} \tag{24a}$$

The amplitude and phase characteristics are

$$|T(j\omega)| = \frac{T}{2}\left|\cot\frac{\omega T}{2}\right| \tag{24b}$$

$$\underline{/T(j\omega)} = -\frac{\pi}{2} \tag{24c}$$

From Eq. (24c) we observe that we have perfect phase characteristics but the amplitude characteristics differ from the ideal characteristics at higher frequencies. Note that the system matrix G_3 in Eq. (22) is constructed by using an odd number of H_j $(j = 1,2,3)$. If an odd number of H_j are used, the approximate models of G_j give ideal phase characteristics and better amplitude characteristics. For example, the z-transform of Equations (4a) and (21) with $G = G_5$ and $M = M_5$ gives

$$Z[\dot{x}(kT)] = AZ[x(kT)] + BZ[u(kT)] = Ax(z) + Bu(z)$$
$$= zx(z) - zx(0) \tag{24d}$$

and

$$Z[x^*(kT + T)] = G_5\, Z[x^*(kT)] + M_5\, Z[u^*(kT)] = G_5 x^*(z) + M_5 \frac{z+1}{2} u(z)$$
$$= zx^*(z) - zx^*(0) \tag{24e}$$

Substituting G_5 and M_5 of Equations (18) and (19) in Eq. (24e) and expressing it in the form of Eq. (24d) yields

$$[I + \frac{1}{12}(AT)^2]\frac{2}{T}\,\frac{z-1}{z+1}\,x^*(z) - [I - \frac{1}{2}AT + \frac{1}{12}(AT)^2]\frac{2}{T}\,\frac{z}{z+1}\,x^*(0)$$
$$= Ax^*(z) + Bu(z) \tag{24f}$$

Comparing Eq. (24d) with (24f) yields

$$Z[\dot{x}(kT)] \cong [I + \frac{1}{12}(AT)^2]\frac{2}{T}\,\frac{z-1}{z+1}\,x(z) - [I - \frac{1}{2}AT + \frac{1}{12}(AT)^2]\frac{2}{T}\,\frac{z}{z+1}\,x(0) \tag{24g}$$

Note that the approximate numerical differentiator in Eq. (23a) differs from that of Eq. (24g) with $x(0) = 0$ by a weighting factor $[I + \frac{1}{12}(AT)^2]$. In the same fashion, if $G = G_7$ and $M = M_7$ are used, then we have

$$Z[\dot{x}(kT)] \cong [I + \frac{1}{60}(AT)^2]^{-1}[I + \frac{1}{10}(AT)^2]\frac{2}{T}\,\frac{z-1}{z+1}\,x(z)$$
$$-[I + \frac{1}{60}(AT)^2]^{-1}[I - \frac{1}{2}AT + \frac{1}{10}(AT)^2 - \frac{1}{120}(AT)^3]\frac{2}{T}\,\frac{z}{z+1}\,x(0) \tag{24h}$$

In general, the approximate numerical differentiator can be written as

$$Z[\dot{x}(kT)] \cong \frac{AT}{2}(N_j - D_j)^{-1}(N_j + D_j)\frac{2}{T}\frac{z-1}{z+1}x(z) - AT(N_j - D_j)^{-1} D_j \frac{2}{T}\frac{z}{z+1}x(0)$$

$$= A(N_j - D_j)^{-1}(N_j + D_j)\frac{z-1}{z+1}x(z) - 2A(N_j - D_j)^{-1}\frac{z}{z+1}x(0) \qquad (24i)$$

where

$$G_j \triangleq D_j^{-1} N_j;\ j = 1,3,5,\ldots$$

The approximate models obtained by using an even number of H_j do not have ideal phase characteristics.

2.5 SAMPLING PERIOD

In a large control system it is often difficult to determine a *minimal common sampling period among* subsystems. This method provides more flexibility in determining the sampling period. In other words, if a more accurate model is used, a larger sampling period can be used.

Consider the commonly used[4-6] system matrix G_3 in Eq. (22) and substitute x = T, or

$$G_3 = [I - \frac{1}{2}Ax]^{-1}[I + \frac{1}{2}Ax] \qquad (25a)$$

To study the relationships between the sampling period of G_3 and that of a more accurate model in Eq. (18), we equate G_3 with G_5 in Eq. (18):

$$[I - \frac{1}{2}Ax]^{-1}[I + \frac{1}{2}Ax] = [I - \frac{1}{2}AT + \frac{1}{12}(AT)^2]^{-1}[I + \frac{1}{2}AT + \frac{1}{12}(AT)^2] \qquad (25b)$$

With no loss of generality, we can assume that A is a diagonal matrix with all eigenvalues λ_j located on the diagonal. The absolute value of the largest eigenvalue, designated λ_m, is used to determine the minimal sampling period. One equation of Eq. (25b) is

$$(1 - \frac{1}{2}\lambda_m x)^{-1}(1 + \frac{1}{2}\lambda_m x) = [1 - \frac{1}{2}\lambda_m T + \frac{1}{12}(\lambda_m T)^2]^{-1}[1 + \frac{1}{2}\lambda_m T + \frac{1}{12}(\lambda_m T)^2] \tag{25c}$$

Solving Eq. (25c) yields

$$x = T\frac{1}{1 + \frac{1}{12}(\lambda_m T)^2} < T \tag{26a}$$

Therefore a larger sampling period can be used if G_5 instead of G_3 is used to approximate the system. In a similar manner, if G_7 were used, then

$$x = T\frac{1 + \frac{1}{60}(\lambda_m T)^2}{1 + \frac{1}{10}(\lambda_m T)^2} < T \tag{26b}$$

Likewise, if G_9 were used, then

$$x = T\frac{1 + \frac{1}{42}(\lambda_m T)^2}{1 + \frac{3}{28}(\lambda_m T)^2 + \frac{1}{1680}(\lambda_m T)^4} < T \tag{26c}$$

Note that the sampling period x in Eq. (25a) is always smaller than the sampling period T in Equations (26a), (26b) or (26c).

2.6 ILLUSTRATIVE EXAMPLE

Consider an unstable continuous-time state equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$x(0) = x_o \tag{27a}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix},\ B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix},\ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and $u(t)$ is the unit-step function. The approximate discrete-time state equation with sampling period $T = \frac{1}{4}$ is required.

$$x^*(k + 1) = Gx^*(k) + Mu(k) \tag{27b}$$

Four approximate models are compared in this example.

Two popular models[2] often used in industry are

$$G = G_3^* = I + AT + \frac{1}{2!}(AT)^2 \tag{28a}$$

$$M = M_3^* = [G_3^* - I]A^{-1}\ B = T[I + \frac{1}{2}\ AT]B \tag{28b}$$

where

$$G_3^* = \begin{bmatrix} 1.46875 & 0.3125 \\ 0.46875 & 0.6875 \end{bmatrix} \text{ and } M_3^* = \begin{bmatrix} 0.625 & 0.0625 \\ 0.3125 & 0.125 \end{bmatrix}$$

and

$$G = G_5^* = I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \frac{1}{4!}(AT)^4 \tag{29a}$$

$$M = M_5^* = T[I + \frac{1}{2!}\ AT + \frac{1}{3!}(AT)^2 + \frac{1}{4!}(AT)^3]B \tag{29b}$$

where

$$G_5^* = \begin{bmatrix} 1.456868 & 0.383138 \\ 0.574707 & 0.499023 \end{bmatrix} \text{ and } H_5^* = \begin{bmatrix} 0.648438 & 0.053060 \\ 0.324219 & 0.165039 \end{bmatrix}$$

Two other models, obtained from matrix continued fractions are

$$G = G_3 = [I - \frac{1}{2} AT]^{-1}[I + \frac{1}{2} AT] \tag{30a}$$

$$M = M_3 = T[I - \frac{1}{2} AT]^{-1} B \tag{30b}$$

where

$$G_3 = \begin{bmatrix} 1.461538 & 0.410256 \\ 0.615384 & 0.435897 \end{bmatrix} \text{ and } M_3 = \begin{bmatrix} 0.666666 & 0.051282 \\ 0.333333 & 0.179487 \end{bmatrix}$$

and

$$G = G_5 = [I - \frac{1}{2} AT + \frac{1}{12}(AT)^2]^{-1}[I + \frac{1}{2} AT + \frac{1}{12}(AT)^2] \tag{31a}$$

$$M = M_5 = T[I - \frac{1}{2} AT + \frac{1}{12}(AT)^2]^{-1} B \tag{31b}$$

where

$$G_5 = \begin{bmatrix} 1.454246 & 0.388804 \\ 0.583206 & 0.482235 \end{bmatrix} \text{ and } M_5 = \begin{bmatrix} 0.648648 & 0.051968 \\ 0.324324 & 0.168417 \end{bmatrix}$$

Note that the model in Eq. (30) is the model used by the referenced authors[4-6].

The exact solution of Eq. (27a) is

$$x_1(t) = \frac{16}{7} e^{2t} - \frac{3}{35} e^{-5t} - \frac{6}{5} \tag{32a}$$

$$x_2(t) = \frac{8}{7} e^{2t} + \frac{9}{35} e^{-5t} - \frac{2}{5} \tag{32b}$$

The responses at the sampling instants k = 0,1,2,3,4 of the exact model and the four approximations are shown in Table 2-1 [state $x_1(kT)$] and Table 2-2 [state $x_2(kT)$]. From the tables we observe that Eq. (31) gives the best approximation.

TABLE 2-1

COMPARISON OF STATE $x_1(kT)$

k	t	EXACT SOL.	DIRECT APPROXIMATION		MATRIX CONT. FRACTION	
		Eq. (32a)	Eq. (28)	Eq. (29)	Eq. (30)	Eq. (31)
0	0.00	1	1	1	1	1
1	0.25	2.544	2.468	2.541	2.589	2.544
2	0.50	5.006	4.811	5.002	5.145	5.006
3	0.75	9.042	8.595	9.036	9.380	9.041
4	1.00	15.689	14.731	15.676	16.436	15.686

TABLE 2-2

COMPARISON OF STATE $x_2(kT)$

k	t	EXACT SOL.	DIRECT APPROXIMATION		MATRIX CONT. FRACTION	
		Eq. (32b)	Eq. (28)	Eq. (29)	Eq. (30)	Eq. (31)
0	0.00	1	1	1	1	1
1	0.25	1.558	1.593	1.563	1.564	1.558
2	0.50	2.728	2.690	2.730	2.788	2.728
3	0.75	4.728	4.542	4.726	4.894	4.728
4	1.00	8.046	7.589	8.041	8.419	8.045

3.0 STABILITY OF COUPLED MULTIVARIABLE MISSILE SYSTEMS

3.1 INTRODUCTION

The accurate description of most practical systems, for example both a small semi-active terminal homing missile system[2] and an aircraft system[12], result in high order coupled multivariable differential equations. Linear representations of these systems are by a set of coupled high-order differential equations or a matrix differential equation. A primary concern in the design of these multivariable systems is the stability problem. One conventional approach is to formulate the system into a high dimensional state equation, then to determine the stability by either directly evaluating the roots of the scalar characteristics polynomial, indirectly applying the Routh criterion[13], or application of Jury's inner theory[14] on the characteristic polynomial. However, the determination of a characteristic polynomial for a large dimensional system is tedious. Moreover, if a system is modeled as a matrix differential equation, it is more natural to determine the stability directly from the matrix polynomial than indirectly from a scalar polynomial. Some approaches have been proposed to determine the stability of a multivariable system directly from the matrix polynomial. Papaconstantinou[15] suggested a scheme for testing stability of polynomial matrices. In his work, a recursive algorithm was developed to compare the normalized largest eigenvalues with unity. However, the method requires the calculation of the eigenvalues of largest moduli. This procedure is difficult due to the problem of convergence of the eigenvalue algorithm. Recently, Shieh and Sacheti[16]

partially extended the scalar Routh criterion[13] to the matrix case. In this work, it is shown that, if a matrix polynomial $B(s) = Is^n + B_n s^{n-1} + \dots + B_1$ is given, a matrix Routh array can be constructed by using the following matrix Routh algorithm:

$$C_{1,j} = B_{n+3-2j} \qquad j = 1,2,3,\dots,\ell$$

$$\text{where } \ell = \begin{cases} \frac{n}{2} + 1 & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{cases}$$

$$C_{2,j} = B_{n+2-2j} \qquad j = 1,2,3,\dots,\ell$$

$$C_{11} = I$$

$$C_{i,j} = C_{i-2,j+1} - H_{i-2}C_{i-1,j+1} \qquad i = 1,2,\dots,\ j = 3,4,\dots$$

$$H_i = C_{i,1}(C_{i+1,1})^{-1} \qquad i = 1,2,\dots,\ n$$

$$\det (C_{i+1,1}) \neq 0 \tag{1}$$

A sufficient condition for stability of the det $[B(s)]$ is that all the "matrix quotients" H_i be real, symmetric, positive definite matrices. Note that this sufficient condition deals only with H_i and not $C_{j,1}$ (the block elements in the first column of the matrix Routh array). Liapunov theory with the state equation in the controllable block companion (controllable phase-variable) form was used to derive their result.

In this report, we develop two approaches for determining the stability of a class of multivariable systems. One approach uses the "matrix quotients" M_j that are developed from an alternate matrix Routh algorithm and a state equation in the observable block companion form[17]. The other approach uses

the block elements in the first column of the matrix Routh array. Two sufficient conditions and three necessary conditions are derived for the stability of matrix polynomials, thereby partially extending the scalar Routh criterion to the matrix Routh criterion.

3.2 SUFFICIENT CONDITIONS

The objective of this report is to establish the criteria for the stability of the following matrix state equations.

$$\sum_{i=1}^{n+1} B_i D^{i-1} x(t) = [0], \quad B_{n+1} = I \tag{2a}$$

and

$$D^{i-1} x(0) = [\alpha_{i-1}] \quad i = 1,2,3,\ldots,n \tag{2b}$$

where $x(t)$ is the m-dimensional state vector. B_i, I, and $[0]$ are m x m real constant matrix, identity matrix and null matrix respectively. For the scalar case, it is well known that a system is asymptotically stable if and only if the Routh array elements in the first column are all positive. Shieh and Sacheti[16] partially extended the Routh criteria[13] to the matrix case and derived a sufficient condition for the stability of a multivariable system in Eq. (2) from the controllable block companion form. In this report we derive some sufficient and some necessary conditions for the system in Eq. (2) from the observable block companion form.

Let us rewrite the system in Eq. (2) into the following observable block companion form:

$$[\dot{x}] = [B][x] \tag{3a}$$

$$[x(0)] = [\alpha] \tag{3b}$$

where

$$[B] = \begin{bmatrix} 0 & 0 & 0 & . & 0 & -B_1 \\ I & 0 & 0 & . & 0 & -B_2 \\ 0 & I & 0 & . & 0 & -B_3 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & I & -B_n \end{bmatrix}$$

The dimensions of the matrix [B], the block elements B_i, and state vector [x] are (nxm) x (nxm), m x m, and (nxm) x 1 respectively. Equation (3) can be transformed into the block Schwarz form by using the following linear transformation:

$$[x] = [K_1][y] \tag{4a}$$

and

$$[\dot{y}] = [K_1]^{-1}[B][K_1][y] = [A][y] \tag{4b}$$

where

$$
[K_1] = \begin{bmatrix}
I & . & D_{n-1,2}D^{-1}_{n-1,1} & . & D_{n-3,3}D^{-1}_{n-3,1} & 0 & D_{n-5,4}D^{-1}_{n-5,1} & 0 & . \\
. & . & . & . & . & . & . & . & . \\
0 & . & I & . & D_{62}D^{-1}_{61} & 0 & D_{43}D^{-1}_{41} & 0 & D_{24}D^{-1}_{21} \\
0 & . & 0 & . & 0 & D_{52}D^{-1}_{51} & 0 & D_{33}D^{-1}_{31} & 0 \\
0 & . & 0 & . & I & 0 & D_{42}D^{-1}_{41} & 0 & D_{23}D^{-1}_{21} \\
0 & . & 0 & . & 0 & I & 0 & D_{32}D^{-1}_{31} & 0 \\
0 & . & 0 & . & 0 & 0 & I & 0 & D_{22}D^{-1}_{21} \\
0 & . & 0 & . & 0 & 0 & 0 & I & 0 \\
0 & . & 0 & . & 0 & 0 & 0 & 0 & I
\end{bmatrix} \tag{4c}
$$

and

$$
[A] = \begin{bmatrix}
0 & -A_1 & 0 & . & 0 & 0 \\
I & 0 & -A_2 & . & 0 & 0 \\
0 & I & 0 & . & 0 & 0 \\
. & . & . & . & . & . \\
0 & 0 & 0 & . & 0 & -A_{n-1} \\
0 & 0 & 0 & . & I & -A_n
\end{bmatrix} \tag{4d}
$$

The dimension of each block element in [A] and $[K_1]$ is m x m. The block elements $D_{i,j}$, having dimension m x m, in Eq. (4c) can be obtained from the following alternate matrix Routh algorithm and alternate matrix Routh array which are different from those in Eq. (1).

Let us define $\ell = \frac{n}{2} + 1$ if n is an even number, otherwise $\ell = \frac{n+1}{2}$, and $D_{i,j}$ as follows:

$$
\begin{aligned}
D_{1,j} &= B_{n+3-2j} \qquad j = 1,2,3,\ldots,\ell \\
D_{2,j} &= B_{n+2-2j} \qquad j = 1,2,3,\ldots,\ell \\
D_{11} &= I
\end{aligned}
\tag{5a}
$$

The alternate matrix Routh array and the matrix Routh algorithm are:

$$
\begin{array}{llllll}
 & D_{11} & D_{12} & D_{13} & D_{14} & \cdot\ \cdot \\
M_1 = D_{21}^{-1}D_{11} \;< & & & & & \\
 & D_{21} & D_{22} & D_{23} & D_{24} & \cdot\ \cdot \\
M_2 = D_{31}^{-1}D_{21} \;< & & & & & \\
 & D_{31} \triangleq D_{12} - D_{22}M_1 & D_{32} \triangleq D_{13} - D_{23}M_1 & D_{33} & & \cdot\ \cdot \\
M_3 = D_{41}^{-1}D_{31} \;< & & & & & \\
 & D_{41} \triangleq D_{22} - D_{32}M_2 & D_{42} \triangleq D_{23} - D_{33}M_2 & D_{43} & & \cdot\ \cdot \\
M_4 = D_{51}^{-1}D_{41} \;< & & & & & \\
 & D_{51} \triangleq D_{32} - D_{42}M_3 & D_{52} & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
 & D_{n,1} & & & & \\
M_n = D_{n+1,1}^{-1}D_{n,1} \;< & & & & & \\
 & D_{n+1,1} & & & &
\end{array}
\tag{5b}
$$

where

$$D_{i,j} = D_{i-2,j+1} - D_{i-1,j+1}M_{i-2} \qquad j = 1,2,\ldots,\ i = 3,4,\ldots$$

$$M_i = D_{i+1,1}^{-1}D_{i,1} \qquad i = 1,2,\ldots,n$$

$$\det\,[D_{i+1,1}] \neq 0 \tag{5c}$$

Shieh and Sacheti[16] have shown that if $H_i = C_{i,1}C_{i+1,1}^{-1}$ for $i = 1,2,\ldots,n$ in Eq. (1) are positive definite, then the system in Eq. (2) is asymptotically stable. Here, we show similar results when replacing H_i by M_i.

<u>Theorem 1</u>:

If $\{M_i\}$ $i = 1,2,\ldots,n$ in Eq. (5) are positive definite, then the system in Eq. (2) is asymptotically stable.

<u>Proof</u>:

Performing the following new transformation

$$[y] = [K_2][z] \tag{6}$$

on Eq. (4) yields

$$\begin{aligned}[\dot{z}] &= [K_2]^{-1}[A][K_2][z] \\ &= [F][z]\end{aligned} \tag{7a}$$

where

$$[K_2] = \begin{bmatrix} D_{n,1} & 0 & . & 0 & 0 \\ 0 & D_{n-1,1} & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & D_{21} & 0 \\ 0 & 0 & . & 0 & D_{11} \end{bmatrix} \tag{7b}$$

and

$$
[F] = \begin{bmatrix}
0 & -M_n^{-1} & 0 & . & 0 & 0 & 0 & 0 \\
M_{n-1}^{-1} & 0 & -M_{n-1}^{-1} & . & 0 & 0 & 0 & 0 \\
0 & M_{n-2}^{-1} & 0 & . & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . \\
0 & 0 & 0 & . & 0 & -M_4^{-1} & 0 & 0 \\
0 & 0 & 0 & . & M_3^{-1} & 0 & -M_3^{-1} & 0 \\
0 & 0 & 0 & . & 0 & M_2^{-1} & 0 & -M_2^{-1} \\
0 & 0 & 0 & . & 0 & 0 & M_1^{-1} & -M_1^{-1}
\end{bmatrix} \tag{7c}
$$

The linear transformation matrix $[K]$ between x coordinates and z coordinates is

$$
[x] = [K][z] = [K_1][K_2][z] \tag{7d}
$$

Now, *consider the following quadratic equation:*

$$
V = [q]^T[P][q] \tag{8a}
$$

where

$$
P = \begin{bmatrix}
M_n & 0 & . & 0 & 0 \\
0 & M_{n-1} & . & 0 & 0 \\
. & . & . & . & . \\
0 & 0 & . & M_2 & 0 \\
0 & 0 & . & 0 & M_1
\end{bmatrix} \tag{8b}
$$

and T in Eq. (8a) designates transpose.

Since $\{M_i\}$ are positive definite which implies that P is positive definite, V is positive definite. The derivative of V is

$$\dot{V} = [q]^T[PF + F^TP][q]$$

$$= -[q]^T[Q][q] \tag{9a}$$

where

$$[Q] = \begin{bmatrix} 0 & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & 2I \end{bmatrix} \tag{9b}$$

From Equations (8) and (9) we can see that V is a Liapunov function. Hence, we conclude that the system in Eq. (2) is asymptotically stable.

From the result obtained in Theorem 1, we establish another sufficient condition for the stability of the system in Eq. (2) by using the block elements $D_{i,1}$ in the matrix Routh array in Eq. (5) instead of the M_i in Eq. (5).

<u>Theorem 2:</u>

If $\{D_{i,1}\}$ $i = 2,4,6,\ldots$, are positive definite, the eigenvalues of $\{D_{i,1}\}$ $i = 1,3,5,\ldots$, are positive and real, and $\{D_{i,1}D_{i+1,1}\}$ $i = 1,3,5,\ldots$, $\{D_{i+1,1}D_{i,1}\}$ $i = 2,4,6,\ldots$, are Hermitian, the system [Eq. (2)] is asymptotically stable.

In order to prove Theorem 2, we need the following lemma which is due to KyFan[18] [P. 137].

Lemma 1. Let K_1 be positive definite and K_2 such that K_1K_2 is Hermitian. Then K_1K_2 is positive definite if and only if the eigenvalues of K_2 are positive and real. In the following lemma, we switch the conditions on K_1 and K_2 yielding the same result.

Lemma 2. Let K_2 be positive definite and K_1 such that K_1K_2 is Hermitian. The K_1K_2 is positive definite if and only if the eigenvalues of K_1 are positive and real.

Proof: Since K_2 is positive definite which implies K_2^T is positive definite, where T designates transpose, it is seen from lemma 1 that $K_2^TK_1^T$ is positive definite if and only if the eigenvalues of K_1^T are positive and real. But $K_2^TK_1^T = (K_1K_2)^T$; i.e., K_1K_2 is positive definite if and only if the eigenvalues of K_1 are positive and real.

Lemma 3. If K_2 is positive definite and K_1K_2 is symmetric, then $K_1^{-1}K_2$ is symmetric.

Proof: Since $(K_1K_2)^T = K_2^TK_1^T = K_2K_1^T = K_1K_2$ which implies $K_1^T = K_2^{-1}K_1K_2$. Hence $(K_1^{-1}K_2)^T = K_2^T(K_1^{-1})^T = K_2K_2^{-1}K_1^{-1}K_2 = K_1^{-1}K_2$; i.e., $K_1^{-1}K_2$ is symmetric.

Proof of Theorem 2:

By lemma 3, we know that $D_{i+1,1}^{-1}D_{i,1}$ is symmetric for $i = 1,2,\ldots$. By lemma 2 or 3, we know that $M_i = D_{i+1,1}^{-1}D_{i,1}$ is positive definite. Hence, the system in Eq. (2) is asymptotically stable following the results of Theorem 1.

In order to show an application of Theorem 1 and Theorem 2, let us consider the following matrix characteristic equation:

$$As^2 + Bs + C = 0 \tag{10a}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix}$$

If we arrange the matrices A, B and C in Eq. (10a) by following the matrix Routh algorithm of Eq. (1), we obtain

$$\begin{array}{lll} & C_{11} = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & C_{12} = C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix} \\ H_1 = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} < & & \\ & C_{21} = B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} & \\ H_2 = \frac{1}{2}\begin{bmatrix} 6 & 0 \\ 2 & \frac{8}{3} \end{bmatrix} < & & \\ & C_{31} = C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix} & \end{array} \tag{10b}$$

In this case, no conclusion can be drawn from the sufficient condition established by Shieh and Sacheti[16]. However, if we arranged the matrices A, B and C according to Eq. (5), we have

$$D_{11} = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D_{12} = C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix}$$

$$M_1 = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} <$$

$$D_{21} = B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$M_2 = \frac{1}{2}\begin{bmatrix} \frac{17}{3} & 1 \\ 1 & 3 \end{bmatrix} <$$

$$D_{31} = C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix} \tag{10c}$$

From Theorem 1, we see that the system is asymptotically stable.

This example shows the application of Theorem 2. Let us consider the following matrix characteristic equation:

$$A_1 S^2 + B_1 S + C_1 = 0 \tag{11a}$$

where

$$D_{11} = A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{21} = B_1 = \begin{bmatrix} 25.77 & 13.7 \\ 13.7 & 7.3 \end{bmatrix}, \; D_{12} = C_1 = D_{31} = \begin{bmatrix} -1 & 2.1 \\ -1 & 2 \end{bmatrix}$$

In Bellman[18] [p.67, p.101] it is shown that, if A_1, B_1 and C_1 are positive definite, then the roots of det $[A_1 S^2 + B_1 S + C_1] = 0$ have negative real parts. But in this example, no conclusion can be made from Bellman's results. However, we know that

$$D_{11} \cdot D_{21} = A_1 \cdot B_1 = \begin{bmatrix} 25.77 & 13.7 \\ 13.7 & 7.3 \end{bmatrix}$$

and

$$D_{31} \cdot D_{21} = C_1 \cdot B_1 = \begin{bmatrix} 3 & 1.63 \\ 1.63 & 0.9 \end{bmatrix} \tag{11b}$$

which are symmetric, B_1 is positive definite, and the eigenvalues of A_1 and C_1 are positive and real. Therefore, from Theorem 2 we conclude that the system in Eq. (11a) is asymptotically stable.

3.3 NECESSARY CONDITIONS

In this section we establish some necessary conditions for the stability of multivariable systems. The failure to satsify the necessary conditions for stability is equivalent to the sufficient conditions for the instability of the same systems; i.e.,

Theorem 3.

If $\{M_i\}$ $i = 1,2,\ldots,n$ are symmetric such that, there exists one $\{M_i\}$ $i = 1,2,\ldots,n$ which is negative definite, negative semi-definite, or indefinite, then the system in Eq. (2) is unstable.

Proof:

Suppose the system is asymptotically stable and one of $\{M_i\}$ is negative definite, negative semi-definite, or indefinite. Since the stability is invariant under the linear transformation and the matrix F in Eq. (7) is a stable matrix. Let us consider the following equation:

$$XF + F^T X = -Q \tag{12}$$

where Q is a matrix defined in Eq. (9b). By Theorem 4 in Bellman[18] [p. 239] we know that Eq. (12) has a unique solution. Since Q is positive semi-definite

we conclude that the solution X of Eq. (12) is also positive semi-definite. It is easy to verify that the matrix P which was defined in Eq. (8b) satisfies Eq. (12). Therefore X = P is positive semi-definite. This implies that at least one of the $\{M_i\}$ is positive semi-definite and others positive definite. This contradicts our assumption that one of the $\{M_i\}$ is negative definite, negative semi-definite, or indefinite. Hence the system in Eq. (2) is unstable if one of the $\{M_i\}$ is negative definite, negative semi-definite, or indefinite.

To show an application of Theorem 3, consider the example[15]:

$$A_1 \frac{d^2y}{dt^2} + B_1 \frac{dy}{dt} + C_1 y = 0 \tag{13a}$$

where

$$A_1 = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 49 & 0 \\ 0 & 49 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -5 & -1 \\ -1 & -10 \end{bmatrix}$$

Applying Eq. (5) yields the matrix Routh array and M_j

$$D_{11} = A_1 = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix} \quad D_{12} = C_1 = \begin{bmatrix} -5 & -1 \\ -1 & -10 \end{bmatrix}$$

$$M_1 = \frac{1}{49}\begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix} <$$

$$D_{21} = B_1 = \begin{bmatrix} 49 & 0 \\ 0 & 49 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} -10 & 1 \\ 1 & -5 \end{bmatrix} <$$

$$D_{31} = C_1 = \begin{bmatrix} -5 & -1 \\ -1 & -10 \end{bmatrix} \tag{13b}$$

M_2 is symmetric and indefinite. According to Theorem 3, the system is unstable.

The following theorem is another criteria for an unstable system.

Theorem 4.

If $D_{11} = B_{n+1} = I$ and the trace of $D_{21} = B_n$ is negative, then the system in Eq. (2) is unstable, where D_{11} and D_{21} are defined in Eq. (5a).

Proof:

The matrix characteristic equation of the system in Eq. (2) is

$$[D(S)] = D_{11}S^n + D_{21}S^{n-1} + D_{12}S^{n-2} + \dots + D_{i,n}$$

$$= B_{n+1}S^n + B_nS^{n-1} + B_{n-1}S^{n-2} + \dots + B_1 \tag{14}$$

where $i = 1$ if n is even and $i = 2$ if n is odd.

Since the sum of the eigenvalues of the system in Eq. (2) is equal to the negative value of the trace of D_{21}, this implies that there exists some eigenvalues of det $[D(S)]$ which are positive. Hence, the system is unstable.

The next criteria is another necessary condition which we state as follows.

Theorem 5.

If det $D_{11} > 0$ and det $D_{i,n} < 0$, or det $D_{11} < 0$ and det $D_{i,n} > 0$, and D_{11}, $D_{i,n}$ are defined in Eq. (14) then the system in Eq. (2) is unstable.

Proof:

Since the system in Eq. (2) has the matrix characteristic equation $[D(S)]$ in Eq. (14), then we expand the det $[D(S)]$. We find the constant term is equal to det B_1 = det $D_{i,n}$. If det $D_{11} > 0$ and det $D_{i,n} < 0$, this implies that the coefficient of the polynomial det $[D(S)]$ has a negative sign. We can then conclude that the det $[D(S)] = 0$ has a solution with a positive real part. Hence the system is unstable.

4.0 CONCLUSIONS

A method, based on a model reduction technique, for accurately representing continuous-time state equations by discrete-time state equations has been presented. Matrix continued fractions have been used as a basis for the derivation. The relationships among commonly used approximation models and the proposed models have been discussed. It has been shown that the models developed by Tustin, Boxer and Thaler and Shieh et.al. are special cases of the methods presented here. Also, it has been shown that if a more accurate model is used, a larger sampling period can be applied. Thus the difficulty of selecting a common sampling period among the subsystems of the T6 missile can be reduced.

Some necessary and sufficient conditions have been developed for the stability of multivariable systems. A linear block transformation has been derived for transforming the coordinates of an observable block companion form to the coordintes of a block Schwarz form. The proposed method has partially extended the scalar Routh criterion to the matrix Routh criterion to a class of multivariable systems.

Thus each subsystem of the T6 missile can be more accurately represented by the discrete-time models proposed in this report. Also, the stability of the coupled yaw and roll system can be determined by the proposed approach. The proposed methods in this report can be applied in general to missile systems of the T6 missile class.

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